Fractional Calculus of Variations in Terms of a Generalized Fractional Integral with Applications to Physics*

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Abstract

We study fractional variational problems in terms of a generalized fractional integral with Lagrangians depending on classical derivatives, generalized fractional integrals and derivatives. We obtain necessary optimality conditions for the basic and isoperimetric problems, as well as natural boundary conditions for free boundary value problems. The fractional action–like variational approach (FALVA) is extended and some applications to Physics discussed.

Keywords: generalized fractional operators; fractional variational analysis; isoperimetric problems; Euler–Lagrange equations; natural boundary conditions; fractional action-like variational approach (FALVA); Caldirola–Kanai oscillatory dynamical systems.

2010 Mathematics Subject Classification: 49K05; 49K21; 49S05; 26A33; 34A08.

1 Introduction

The calculus of variations is a beautiful and useful field of mathematics that deals with problems of determining extrema (maxima or minima) of functionals [39, 40, 58]. It starts with the simplest problem of finding a function extremizing (minimizing or maximizing) an integral

$$\mathcal{J}(y) = \int_a^b F(t, y(t), y'(t))dt$$

subject to boundary conditions $y(a) = y_a$ and $y(b) = y_b$. In the literature many generalizations of this problem were proposed, including problems with multiple integrals, functionals containing higher-order derivatives, and functionals depending on several functions [35, 37, 45]. Of our interest is an extension proposed by Riewe in 1996-1997, where fractional derivatives (real or complex order) are introduced in the Lagrangian [50, 51].

During the last decade, fractional problems have increasingly attracted the attention of many researchers. As mentioned in [9], Science Watch of Thomson Reuters identified the subject as an *Emerging Research Front* area. Fractional derivatives are nonlocal operators and are historically applied in the study of nonlocal or time dependent processes [46]. The first and well established

*Part of the first author's Ph.D., which is carried out at the University of Aveiro under the Doctoral Programme *Mathematics and Applications* of Universities of Aveiro and Minho. Submitted 01/Jan/2012; revised 25/Feb/2012; accepted 27/Feb/2012; for publication in *Abstract and Applied Analysis* (http://www.hindawi.com/journals/aaa/).

application of fractional calculus in Physics was in the framework of anomalous diffusion, which is related to features observed in many physical systems. Here we can mention the report [38] demonstrating that fractional equations works as a complementary tool in the description of anomalous transport processes. Within the fractional approach it is possible to include external fields in a straightforward manner. As a consequence, in a short period of time the list of applications expanded. Applications include chaotic dynamics [60], material sciences [33], mechanics of fractal and complex media [12,32], quantum mechanics [25,31], physical kinetics [61], long-range dissipation [54], long-range interaction [53, 55], just to mention a few. One of the most remarkable applications of fractional calculus appears, however, in the fractional variational calculus, in the context of classical mechanics. Riewe [50, 51] shows that a Lagrangian involving fractional time derivatives leads to an equation of motion with nonconservative forces such as friction. It is a remarkable result since frictional and nonconservative forces are beyond the usual macroscopic variational treatment and, consequently, beyond the most advanced methods of classical mechanics [30]. Riewe generalizes the usual variational calculus, by considering Lagrangians that dependent on fractional derivatives, in order to deal with nonconservative forces. Recently, several different approaches have been developed to generalize the least action principle and the Euler–Lagrange equations to include fractional derivatives. Results include problems depending on Caputo fractional derivatives, Riemann–Liouville fractional derivatives and others [3,4,10,11,14,19,20,34,36,41–43,52].

A more general unifying perspective to the subject is, however, possible, by considering fractional operators depending on general kernels [1,28,44]. In this work we follow such an approach, developing a generalized fractional calculus of variations. We consider very general problems, where the classical integrals are substituted by generalized fractional integrals, and the Lagrangians depend not only on classical derivatives but also on generalized fractional operators. Problems of the type considered here, for particular kernels, are important in Physics [18]. Here we obtain general necessary optimality conditions, for several types of variational problems, which are valid for rather arbitrary operators and kernels. By choosing particular operators and kernels, one obtains the recent results available in the literature of Mathematical Physics [8,15–18,24].

The paper is organized as follows. In Section 2 we introduce the generalized fractional operators and prove some of its basic properties. Section 3 is dedicated to prove integration by parts formulas for the generalized fractional operators. Such formulas are then used in later sections to prove necessary optimality conditions (Theorems 14 and 23). In Sections 4, 5 and 6 we study three important classes of generalized variational problems: we obtain fractional Euler–Lagrange conditions for the fundamental (Section 4) and generalized isoperimetric problems (Section 6), as well as fractional natural boundary conditions for generalized free-boundary value problems (Section 5). Finally, two illustrative examples are discussed in detail in Section 7, while applications to Physics are given in Section 8: in Section 8.1 we obtain the damped harmonic oscillator in quantum mechanics, in Section 8.2 we show how results from FALVA Physics can be obtained. We end with Section 9 of conclusion, pointing out an important direction of future research.

2 Preliminaries

In this section we present definitions and properties of generalized fractional operators. As particular cases, by choosing appropriate kernels, these operators are reduced to standard fractional integrals and fractional derivatives. Other nonstandard kernels can also be considered as particular cases. For more on the subject of generalized fractional calculus and applications, we refer the reader to the book [28]. Throughout the text, α denotes a real number between zero and one. Following [5], we use round brackets for the arguments of functions, and square brackets for the arguments of operators. By definition, an operator receives and returns a function.

Definition 1 (Generalized fractional integral)**.** *The operator K_P^α is given by*

$$K_P^\alpha\left[f\right](x) = K_P^\alpha\left[t \mapsto f(t)\right](x) = p\int\limits_a^x k_\alpha(x,t)f(t)dt + q\int\limits_x^b k_\alpha(t,x)f(t)dt,$$

where $P = \langle a, x, b, p, q\rangle$ is the parameter set *(p-set for brevity), $x \in [a,b]$, p, q are real numbers, and $k_\alpha(x,t)$ is a kernel which may depend on α. The operator K_P^α is referred as the* operator K *(K-op for simplicity) of order α and p-set P, while $K_P^\alpha[f]$ is called the* operation K *(or K-opn) of f of order α and p-set P.*

Note that if we define

$$G(x,t) := \begin{cases} pk_\alpha(x,t) & \text{if } t < x, \\ qk_\alpha(t,x) & \text{if } t \geq x, \end{cases}$$

then the operator K_P^α can be written in the form

$$K_P^\alpha\left[f\right](x) = K_P^\alpha\left[t \mapsto f(t)\right](x) = \int_a^b G(x,t)f(t)dt.$$

This is a particular case of one of the oldest and most respectable class of operators, so called Fredholm operators [23, 47].

Theorem 2 (cf. Example 6 of [23])**.** *Let $\alpha \in (0,1)$ and $P = \langle a, x, b, p, q\rangle$. If k_α is a square integrable function on the square $\Delta = [a,b] \times [a,b]$, then $K_P^\alpha : L_2\left([a,b]\right) \to L_2\left([a,b]\right)$ is well defined, linear, and bounded operator.*

Theorem 3. *Let k_α be a difference kernel, i.e., let $k_\alpha \in L_1\left([a,b]\right)$ with $k_\alpha(x,t) = k_\alpha(x-t)$. Then, $K_P^\alpha : L_1\left([a,b]\right) \to L_1\left([a,b]\right)$ is a well defined bounded and linear operator.*

Proof. Obviously, the operator is linear. Let $\alpha \in (0,1)$, $P = \langle a, t, b, p, q\rangle$, and $f \in L_1\left([a,b]\right)$. Define

$$F(\tau,t) := \begin{cases} |pk_\alpha(t-\tau)| \cdot |f(\tau)| & \text{if } \tau \leq t \\ |qk_\alpha(\tau-t)| \cdot |f(\tau)| & \text{if } \tau > t \end{cases}$$

for all $(\tau,t) \in \Delta = [a,b] \times [a,b]$. Since F is measurable on the square Δ, we have

$$\begin{aligned} \int_a^b \left(\int_a^b F(\tau,t)dt\right)d\tau &= \int_a^b \left[|f(\tau)|\left(\int_\tau^b |pk_\alpha(t-\tau)|\,dt + \int_a^\tau |qk_\alpha(\tau-t)|\,dt\right)\right]d\tau \\ &\leq \int_a^b |f(\tau)|\,\big||p|-|q|\big|\,\|k_\alpha\|\,d\tau \\ &= \big||p|-|q|\big| \cdot \|k_\alpha\| \cdot \|f\|. \end{aligned}$$

It follows from Fubini's theorem that F is integrable on the square Δ. Moreover,

$$\begin{aligned} \|K_P^\alpha[f]\| &= \int_a^b \left|p\int_a^t k_\alpha(t-\tau)f(\tau)d\tau + q\int_t^b k_\alpha(\tau-t)f(\tau)d\tau\right|dt \\ &\leq \int_a^b \left(|p|\int_a^t |k_\alpha(t-\tau)| \cdot |f(\tau)|\,d\tau + |q|\int_t^b |k_\alpha(\tau-t)| \cdot |f(\tau)|\,d\tau\right)dt \\ &= \int_a^b \left(\int_a^b F(\tau,t)d\tau\right)dt \\ &\leq \big||p|-|q|\big| \cdot \|k_\alpha\| \cdot \|f\|. \end{aligned}$$

Hence, $K_P^\alpha : L_1\left([a,b]\right) \to L_1\left([a,b]\right)$ and $\|K_P^\alpha\| \leq \big||p|-|q|\big| \cdot \|k_\alpha\|$. □

Remark 4. *The K-op reduces to the left and the right Riemann–Liouville fractional integrals from a suitably chosen kernel $k_\alpha(x,t)$ and p-set P. Let $k_\alpha(x,t) = k_\alpha(x-t) = \frac{1}{\Gamma(\alpha)}(x-t)^{\alpha-1}$:*

- *if $P = \langle a, x, b, 1, 0\rangle$, then*

$$K_P^\alpha[f](x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt =: {}_aI_x^\alpha[f](x)$$

  is the standard left Riemann–Liouville fractional integral of f of order α;

- *if $P = \langle a, x, b, 0, 1\rangle$, then*

$$K_P^\alpha[f](x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt =: {}_xI_b^\alpha[f](x)$$

  is the standard right Riemann–Liouville fractional integral of f of order α.

Corollary 5. *Operators ${}_aI_x^\alpha, {}_xI_b^\alpha : L_1([a,b]) \to L_1([a,b])$ are well defined, linear and bounded.*

The generalized fractional derivatives A_P^α and B_P^α are defined in terms of the generalized fractional integral K-op.

Definition 6 (Generalized Riemann–Liouville fractional derivative)**.** *Let P be a given parameter set and $0 < \alpha < 1$. The operator A_P^α is defined by $A_P^\alpha = D \circ K_P^{1-\alpha}$, where D denotes the standard derivative operator, and is referred as the* operator A *(A-op) of order α and p-set P, while $A_P^\alpha[f]$, for a function f such that $K_P^{1-\alpha}[f] \in AC([a,b])$, is called the* operation A *(A-opn) of f of order α and p-set P.*

Definition 7 (Generalized Caputo fractional derivative)**.** *Let P be a given parameter set and $\alpha \in (0,1)$. The operator B_P^α is defined by $B_P^\alpha = K_P^{1-\alpha} \circ D$, where D denotes the standard derivative operator, and is referred as the* operator B *(B-op) of order α and p-set P, while $B_P^\alpha[f]$, for a function $f \in AC([a,b])$, is called the* operation B *(B-opn) of f of order α and p-set P.*

Remark 8. *The standard Riemann–Liouville and Caputo fractional derivatives are easily obtained from the generalized operators A_P^α and B_P^α, respectively. Let $k_{1-\alpha}(x,t) = k_{1-\alpha}(x-t) = \frac{(x-t)^{-\alpha}}{\Gamma(1-\alpha)}$:*

- *if $P = \langle a, x, b, 1, 0\rangle$, then*

$$A_P^\alpha[f](x) = \frac{1}{\Gamma(1-\alpha)} D\left[\xi \mapsto \int_a^\xi (\xi-t)^{-\alpha} f(t)dt\right](x) =: {}_aD_x^\alpha[f](x)$$

  is the standard left Riemann–Liouville fractional derivative of f of order α, while

$$B_P^\alpha[f](x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{-\alpha} D[f](t)dt =: {}_a^C D_x^\alpha[f](x)$$

  is the standard left Caputo fractional derivative of f of order α;

- *if $P = \langle a, x, b, 0, 1\rangle$, then*

$$-A_P^\alpha[f](x) = \frac{-1}{\Gamma(1-\alpha)} D\left[\xi \mapsto \int_\xi^b (t-\xi)^{-\alpha} f(t)dt\right](x) =: {}_xD_b^\alpha[f](x)$$

  is the standard right Riemann–Liouville fractional derivative of f of order α, while

$$-B_P^\alpha[f](x) = \frac{-1}{\Gamma(1-\alpha)} \int_x^b (t-x)^{-\alpha} D[f](t)dt =: {}_x^C D_b^\alpha[f](x)$$

  is the standard right Caputo fractional derivative of f of order α.

3 On generalized fractional integration by parts

We now prove integration by parts formulas for generalized fractional operators.

Theorem 9 (Fractional integration by parts for the K-op)**.** *Let $\alpha \in (0,1)$, $P = \langle a, t, b, p, q \rangle$, k_α be a square-integrable function on $\Delta = [a,b] \times [a,b]$, and $f, g \in L_2([a,b])$. The generalized fractional integral K_P^α satisfies the integration by parts formula*

$$\int\limits_a^b g(x) K_P^\alpha [f](x) dx = \int\limits_a^b f(x) K_{P^*}^\alpha [g](x) dx, \tag{1}$$

where $P^* =< a, t, b, q, p >$.

Proof. Define

$$F(\tau, t) := \begin{cases} |pk_\alpha(t,\tau)| \cdot |g(t)| \cdot |f(\tau)| & \text{if } \tau \le t \\ |qk_\alpha(\tau,t)| \cdot |g(t)| \cdot |f(\tau)| & \text{if } \tau > t \end{cases}$$

for all $(\tau, t) \in \Delta$. Applying Holder's inequality, we obtain

$$\begin{aligned}
\int_a^b \left(\int_a^b F(\tau,t) dt \right) d\tau &= \int_a^b \left[|f(\tau)| \left(\int_\tau^b |pk_\alpha(t,\tau)| \cdot |g(t)| \, dt + \int_a^\tau |qk_\alpha(\tau,t)| \cdot |g(t)| \, dt \right) \right] d\tau \\
&\le \int_a^b \left[|f(\tau)| \left(\int_a^b |pk_\alpha(t,\tau)| \cdot |g(t)| \, dt + \int_a^b |qk_\alpha(\tau,t)| \cdot |g(t)| \, dt \right) \right] d\tau \\
&\le \int_a^b \left\{ |f(\tau)| \left[\left(\int_a^b |pk_\alpha(t,\tau)|^2 \, dt \right)^{\frac{1}{2}} \left(\int_a^b |g(t)|^2 \, dt \right)^{\frac{1}{2}} \right. \right. \\
&\qquad \left. \left. + \left(\int_a^b |qk_\alpha(\tau,t)|^2 \, dt \right)^{\frac{1}{2}} \left(\int_a^b |g(t)|^2 \, dt \right)^{\frac{1}{2}} \right] \right\} d\tau.
\end{aligned}$$

By Fubini's theorem, functions $k_{\alpha,\tau}(t) := k_\alpha(t,\tau)$ and $\hat{k}_{\alpha,\tau}(t) := k_\alpha(\tau,t)$ belong to $L_2([a,b])$ for almost all $\tau \in [a,b]$. Therefore,

$$\begin{aligned}
&\int_a^b \left\{ |f(\tau)| \left[\left(\int_a^b |pk_\alpha(t,\tau)|^2 \, dt \right)^{\frac{1}{2}} \left(\int_a^b |g(t)|^2 \, dt \right)^{\frac{1}{2}} \right. \right. \\
&\qquad \left. \left. + \left(\int_a^b |qk_\alpha(\tau,t)|^2 \, dt \right)^{\frac{1}{2}} \left(\int_a^b |g(t)|^2 \, dt \right)^{\frac{1}{2}} \right] \right\} d\tau \\
&= \|g\|_2 \int_a^b \left[|f(\tau)| \left(\|pk_{\alpha,\tau}\|_2 + \left\| q\hat{k}_{\alpha,\tau} \right\|_2 \right) \right] d\tau \\
&\le \|g\|_2 \left(\int_a^b |f(\tau)|^2 \, d\tau \right)^{\frac{1}{2}} \left(\int_a^b \left| \|pk_{\alpha,\tau}\|_2 + \left\| q\hat{k}_{\alpha,\tau} \right\|_2 \right|^2 d\tau \right)^{\frac{1}{2}} \\
&\le \|g\|_2 \cdot \|f\|_2 \left(\|pk_\alpha\|_2 + \|qk_\alpha\|_2 \right) < \infty.
\end{aligned}$$

Hence, we can use again Fubini's theorem to change the order of integration:

$$\begin{aligned}\int_a^b g(t)K_P^\alpha[f](t)dt &= p\int_a^b g(t)dt\int_a^t f(\tau)k_\alpha(t,\tau)d\tau + q\int_a^b g(t)dt\int_t^b f(\tau)k_\alpha(\tau,t)d\tau\\ &= p\int_a^b f(\tau)d\tau\int_\tau^b g(t)k_\alpha(t,\tau)dt + q\int_a^b f(\tau)d\tau\int_a^\tau g(t)k_\alpha(\tau,t)dt\\ &= \int_a^b f(\tau)K_{P^*}^\alpha[g](\tau)d\tau.\end{aligned}$$

□

Theorem 10. *Let $0<\alpha<1$ and $P=\langle a,x,b,p,q\rangle$. If $k_\alpha(x,t)=k_\alpha(x-t)$, $k_\alpha, f\in L_1([a,b])$, and $g\in C([a,b])$, then the operator K_P^α satisfies the integration by parts formula* (1).

Proof. Define

$$F(t,x):=\begin{cases}|pk_\alpha(x-t)|\cdot|g(x)|\cdot|f(t)| & \text{if } t\le x\\ |qk_\alpha(t-x)|\cdot|g(x)|\cdot|f(t)| & \text{if } t>x\end{cases}$$

for all $(t,x)\in\Delta=[a,b]\times[a,b]$. Since g is a continuous function on $[a,b]$, it is bounded on $[a,b]$, i.e., there exists a real number $C>0$ such that $|g(x)|\le C$ for all $x\in[a,b]$. Therefore,

$$\begin{aligned}\int_a^b\left(\int_a^b F(t,x)dt\right)dx &= \int_a^b\left[|f(t)|\left(\int_t^b |pk_\alpha(x-t)|\cdot|g(x)|\,dx+\int_a^t |qk_\alpha(t-x)|\cdot|g(x)|\,dx\right)\right]dt\\ &\le \int_a^b\left[|f(t)|\left(\int_a^b |pk_\alpha(x-t)|\cdot|g(x)|\,dx+\int_a^b |qk_\alpha(t-x)|\cdot|g(x)|\,dx\right)\right]dt\\ &\le C\int_a^b\left[|f(t)|\left(\int_a^b |pk_\alpha(x-t)|\,dx+\int_a^b |qk_\alpha(t-x)|\,dx\right)\right]dt\\ &= C\big||p|-|q|\big|\,\|k_\alpha\|\,\|f\|<\infty.\end{aligned}$$

Hence, we can use Fubini's theorem to change the order of integration in iterated integrals. □

Theorem 11 (Generalized fractional integration by parts)**.** *Let $\alpha\in(0,1)$ and $P=\langle a,t,b,p,q\rangle$. If functions $f, K_{P^*}^{1-\alpha}[g]\in AC([a,b])$, and we are in conditions to use formula* (1) *(Theorem 9 or Theorem 10), then*

$$\int_a^b g(x)B_P^\alpha[f](x)dx = f(x)K_{P^*}^{1-\alpha}[g](x)\Big|_a^b-\int_a^b f(x)A_{P^*}^\alpha[g](x)dx, \tag{2}$$

where $P^*=<a,t,b,q,p>$.

Proof. From Definition 7 we know that $B_P^\alpha[f](x)=K_P^{1-\alpha}[D[f]](x)$. It follows that

$$\int_a^b g(x)B_P^\alpha[f](x)dx=\int_a^b g(x)K_P^{1-\alpha}[D[f]](x)dx.$$

By relation (1)

$$\int_a^b g(x)B_P^\alpha[f](x)dx=\int_a^b D[f](x)K_{P^*}^{1-\alpha}[g](x)dx,$$

and the standard integration by parts formula implies (2):

$$\int_a^b g(x)B_P^\alpha[f](x)dx = f(x)K_{P^*}^{1-\alpha}[g](x)\Big|_a^b-\int_a^b f(x)D\left[K_{P^*}^{1-\alpha}[g]\right](x)dx.$$

□

Corollary 12 (cf. [29]). *Let $0 < \alpha < 1$. If $f, {}_xI_b^{1-\alpha}[g] \in AC([a,b])$, then*

$$\int_a^b g(x)\, {}_a^C D_x^\alpha [f](x)dx = f(x) {}_xI_b^{1-\alpha}[g](x)\Big|_{x=a}^{x=b} + \int_a^b f(x) {}_xD_b^\alpha [g](x)dx.$$

4 The generalized fundamental variational problem

By $\partial_i F$ we denote the partial derivative of a function F with respect to its ith argument. We consider the problem of finding a function $y = t \mapsto y(t)$, $t \in [a,b]$, that gives an extremum (minimum or maximum) to the functional

$$\mathcal{J}(y) = K_{P_1}^\alpha \left[t \mapsto F\left(t, y(t), y'(t), B_{P_2}^\beta [y](t), K_{P_3}^\gamma [y](t)\right)\right](b) \tag{3}$$

when subject to the boundary conditions

$$y(a) = y_a, \quad y(b) = y_b, \tag{4}$$

where $\alpha, \beta, \gamma \in (0,1)$, $P_1 =< a, b, b, 1, 0 >$ and $P_j =< a, t, b, p_j, q_j >$, $j = 2,3$. For simplicity of notation we introduce the operator $\{\cdot\}_{P_2,P_3}^{\beta,\gamma}$ defined by

$$\{y\}_{P_2,P_3}^{\beta,\gamma}(t) = \left(t, y(t), y'(t), B_{P_2}^\beta[\tau \mapsto y(\tau)](t), K_{P_3}^\gamma[\tau \mapsto y(\tau)](t)\right).$$

With the new notation one can write (3) simply as $\mathcal{J}(y) = K_{P_1}^\alpha\left[F\{y\}_{P_2,P_3}^{\beta,\gamma}\right](b)$. The operator $K_{P_1}^\alpha$ has kernel $k_\alpha(x,t)$, and operators $B_{P_2}^\beta$ and $K_{P_3}^\gamma$ have kernels $h_{1-\beta}(t,\tau)$ and $h_\gamma(t,\tau)$, respectively. In the sequel we assume that:

(H1) Lagrangian $F \in C^1\left([a,b] \times \mathbb{R}^4; \mathbb{R}\right)$;

(H2) functions $A_{P_2^*}^\beta\left[\tau \mapsto k_\alpha(b,\tau)\partial_4 F\{y\}_{P_2,P_3}^{\beta,\gamma}(\tau)\right]$, $K_{P_3^*}^\gamma\left[\tau \mapsto k_\alpha(b,\tau)\partial_5 F\{y\}_{P_2,P_3}^{\beta,\gamma}(\tau)\right]$, $D\left[t \mapsto \partial_3 F\{y\}_{P_2,P_3}^{\beta,\gamma}(t)k_\alpha(b,t)\right]$ and $t \mapsto k_\alpha(b,t)\partial_2 F\{y\}_{P_2,P_3}^{\beta,\gamma}(t)$ are continuous on (a,b);

(H3) functions $t \mapsto \partial_3 F\{y\}_{P_2,P_3}^{\beta,\gamma}(t)k_\alpha(b,t)$, $K_{P_2^*}^{1-\beta}\left[\tau \mapsto k_\alpha(b,\tau)\partial_4 F\{y\}_{P_2,P_3}^{\beta,\gamma}(\tau)\right] \in AC([a,b])$;

(H4) kernels $k_\alpha(x,t)$, $h_{1-\beta}(t,\tau)$ and $h_\gamma(t,\tau)$ are such that we are in conditions to use Theorems 9, 10 and 11.

Definition 13. *A function $y \in C^1([a,b];\mathbb{R})$ is said to be admissible for the fractional variational problem* (3)–(4)*, if functions $B_{P_2}^\beta[y]$ and $K_{P_3}^\gamma[y]$ exist and are continuous on the interval $[a,b]$, and y satisfies the given boundary conditions* (4).

Theorem 14. *If y is a solution to problem* (3)–(4)*, then y satisfies the generalized Euler–Lagrange equation*

$$\begin{aligned} k_\alpha(b,t)\partial_2 F\{y\}_{P_2,P_3}^{\beta,\gamma}(t) - \frac{d}{dt}\left(\partial_3 F\{y\}_{P_2,P_3}^{\beta,\gamma}(t)k_\alpha(b,t)\right) \\ - A_{P_2^*}^\beta\left[\tau \mapsto k_\alpha(b,\tau)\partial_4 F\{y\}_{P_2,P_3}^{\beta,\gamma}(\tau)\right](t) + K_{P_3^*}^\gamma\left[\tau \mapsto k_\alpha(b,\tau)\partial_5 F\{y\}_{P_2,P_3}^{\beta,\gamma}(\tau)\right](t) = 0 \end{aligned} \tag{5}$$

for all $t \in (a,b)$.

Proof. Suppose that y is an extremizer of $\mathcal{J}$. Consider the value of $\mathcal{J}$ at a nearby function $\hat{y} = y + \varepsilon\eta$, where $\varepsilon \in \mathbb{R}$ is a small parameter, and $\eta \in C^1([a,b];\mathbb{R})$ is an arbitrary function with continuous B-op and K-op. We require that $\eta(a) = \eta(b) = 0$. Let

$$\begin{aligned} \mathcal{J}(\hat{y}) = J(\varepsilon) &= K_{P_1}^\alpha\left[t \mapsto F\left(t, \hat{y}(t), \hat{y}'(t), B_{P_2}^\beta[\hat{y}](t), K_{P_3}^\gamma[\hat{y}](t)\right)\right](b) \\ &= \int_a^b k_\alpha(b,t) F\left(t, y(t) + \varepsilon\eta(t), \frac{d}{dt}\left(y(t) + \varepsilon\eta(t)\right), B_{P_2}^\beta[y + \varepsilon\eta](t), K_{P_3}^\gamma[y + \varepsilon\eta](t)\right) dt. \end{aligned}$$

A necessary condition for y to be an extremizer is given by

$$\begin{aligned}
\left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0} = 0 \Leftrightarrow K_{P_1}^{\alpha}\Big[&\partial_2 F\{y\}_{P_2,P_3}^{\beta,\gamma}\eta + \partial_3 F\{y\}_{P_2,P_3}^{\beta,\gamma} D[\eta] \\
&+ \partial_4 F\{y\}_{P_2,P_3}^{\beta,\gamma} B_{P_2}^{\beta}[\eta] + \partial_5 F\{y\}_{P_2,P_3}^{\beta,\gamma} K_{P_3}^{\gamma}[\eta]\Big](b) = 0 \\
\Leftrightarrow \int\limits_a^b \Bigg(&\partial_2 F\{y\}_{P_2,P_3}^{\beta,\gamma}(t)\eta(t) + \partial_3 F\{y\}_{P_2,P_3}^{\beta,\gamma}(t)\frac{d}{dt}\eta(t) \\
&+ \partial_4 F\{y\}_{P_2,P_3}^{\beta,\gamma}(t) B_{P_2}^{\beta}[\eta](t) + \partial_5 F\{y\}_{P_2,P_3}^{\beta,\gamma}(t) K_{P_3}^{\gamma}[\eta](t)\Bigg) k_\alpha(b,t)dt = 0.
\end{aligned} \tag{6}$$

Using classical and generalized fractional integration by parts formulas (Theorems 9, 10 and 11),

$$\begin{aligned}
&\int_a^b \partial_3 F\{y\}_{P_2,P_3}^{\beta,\gamma}(t) k_\alpha(b,t)\frac{d}{dt}\eta(t)dt \\
&\qquad = \partial_3 F\{y\}_{P_2,P_3}^{\beta,\gamma}(t) k_\alpha(b,t)\eta(t)\Big|_a^b - \int_a^b \frac{d}{dt}\left(\partial_3 F\{y\}_{P_2,P_3}^{\beta,\gamma}(t) k_\alpha(b,t)\right)\eta(t)dt,
\end{aligned}$$

$$\begin{aligned}
&\int\limits_a^b \partial_4 F\{y\}_{P_2,P_3}^{\beta,\gamma}(t) k_\alpha(b,t) B_{P_2}^{\beta}[\eta](t)dt \\
&= K_{P_2^*}^{1-\beta}\left[\tau \mapsto k_\alpha(b,\tau)\partial_4 F\{y\}_{P_2,P_3}^{\beta,\gamma}(\tau)\right](t)\eta(t)\Big|_a^b - \int\limits_a^b A_{P_2^*}^{\beta}\left[\tau \mapsto k_\alpha(b,\tau)\partial_4 F\{y\}_{P_2,P_3}^{\beta,\gamma}(\tau)\right](t)\,\eta(t)dt
\end{aligned}$$

and

$$\int\limits_a^b k_\alpha(b,t)\partial_5 F\{y\}_{P_2,P_3}^{\beta,\gamma}(t) K_{P_3}^{\gamma}[\eta](t)dt = \int\limits_a^b K_{P_3^*}^{\gamma}\left[\tau \mapsto k_\alpha(b,\tau)\partial_5 F\{y\}_{P_2,P_3}^{\beta,\gamma}(\tau)\right](t)\,\eta(t)dt,$$

where $P_j^* =< a, t, b, q_j, p_j >$, $j = 2, 3$. Because $\eta(a) = \eta(b) = 0$, (6) simplifies to

$$\begin{aligned}
&\int_a^b \Bigg\{ k_\alpha(b,t)\partial_2 F\{y\}_{P_2,P_3}^{\beta,\gamma}(t) - \frac{d}{dt}\left(\partial_3 F\{y\}_{P_2,P_3}^{\beta,\gamma}(t) k_\alpha(b,t)\right) \\
&- A_{P_2^*}^{\beta}\left[\tau \mapsto k_\alpha(b,\tau)\partial_4 F\{y\}_{P_2,P_3}^{\beta,\gamma}(\tau)\right](t) + K_{P_3^*}^{\gamma}\left[\tau \mapsto k_\alpha(b,\tau)\partial_5 F\{y\}_{P_2,P_3}^{\beta,\gamma}(\tau)\right](t)\Bigg\}\eta(t)dt = 0.
\end{aligned}$$

We obtain (5) by application of the fundamental lemma of the calculus of variations (see, e.g., [22, Section 2.2]). □

The next corollary gives an extension of the main result of [19].

Corollary 15. *If y is a solution to the problem of minimizing or maximizing*

$$\mathcal{J}(y) = {}_aI_b^{\alpha}\left[t \mapsto F\left(t, y(t), y'(t), {}_a^CD_t^{\beta}[y](t)\right)\right](b) \tag{7}$$

in the class $y \in C^1([a,b];\mathbb{R})$ subject to the boundary conditions

$$y(a) = y_a, \quad y(b) = y_b, \tag{8}$$

where $\alpha, \beta \in (0,1)$, $F \in C^1\left([a,b] \times \mathbb{R}^3; \mathbb{R}\right)$ *and* $\tau \mapsto (b-\tau)^{\alpha-1}\partial_4 F\left(\tau, y(\tau), y'(\tau), {}^C_a D^\beta_\tau\left[y\right](\tau)\right)$ *has continuous Riemann-Liouville fractional derivative* ${}_tD^\beta_b$, *then*

$$\partial_2 F\left(t, y(t), y'(t), {}^C_a D^\beta_t\left[y\right](t)\right) \cdot (b-t)^{\alpha-1} - \frac{d}{dt}\left\{\partial_3 F\left(t, y(t), y'(t), {}^C_a D^\beta_t\left[y\right](t)\right) \cdot (b-t)^{\alpha-1}\right\} + {}_tD^\beta_b\left[\tau \mapsto (b-\tau)^{\alpha-1}\partial_4 F\left(\tau, y(\tau), y'(\tau), {}^C_a D^\beta_\tau\left[y\right](\tau)\right)\right](t) = 0 \quad (9)$$

for all $t \in (a,b)$.

Proof. Choose $k_\alpha(x,t) = \frac{1}{\Gamma(\alpha)}(x-t)^{\alpha-1}$, $h_{1-\beta}(t,\tau) = \frac{1}{\Gamma(1-\beta)}(t-\tau)^{-\beta}$, and $P_2 =< a, t, b, 1, 0 >$. Then the K-op, the A-op and the B-op reduce to the left fractional integral, the left Riemann-Liouville and the left Caputo fractional derivatives, respectively. Therefore, problem (7)–(8) is a particular case of problem (3)–(4) and (9) follows from (5) with $\partial_5 F = 0$. □

The following result is the Caputo analogous to the main result of [7] done for the Riemann–Liouville fractional derivative.

Corollary 16. *Let* $\beta, \gamma \in (0,1)$. *If* y *is a solution to the problem*

$$\int_a^b F\left(t, y(t), y'(t), {}^C_a D^\beta_t[y](t), {}_aI^\gamma_t[y](t)\right) dt \longrightarrow extr$$

$$y \in C^1\left([a,b]; \mathbb{R}\right)$$
$$y(a) = y_a, \quad y(b) = y_b,$$

then

$$\partial_2 F\left(t, y(t), y'(t), {}^C_a D^\beta_t[y](t), {}_aI^\gamma_t[y](t)\right) - \frac{d}{dt}\partial_3 F\left(t, y(t), y'(t), {}^C_a D^\beta_t[y](t), {}_aI^\gamma_t[y](t)\right) + {}_tD^\beta_b\left[\tau \mapsto \partial_4 F\left(\tau, y(\tau), y'(\tau), {}^C_a D^\beta_\tau[y](\tau), {}_aI^\gamma_\tau[y](\tau)\right)\right](t) + {}_tI^\beta_b\left[\tau \mapsto \partial_5 F\left(\tau, y(\tau), y'(\tau), {}^C_a D^\beta_\tau[y](\tau), {}_aI^\gamma_\tau[y](\tau)\right)\right](t) = 0 \quad (10)$$

holds for all $t \in [a,b]$.

Proof. The Euler–Lagrange equation (10) follows from (5) by choosing p-sets $P_1 =< a, b, b, 1, 0 >$, $P_2 = P_3 =< a, t, b, 1, 0 >$, and kernels $k_\alpha(x,t) = 1$, $h_{1-\beta}(t,\tau) = \frac{1}{\Gamma(1-\beta)}(t-\tau)^{-\beta}$, and $h_\gamma(t,\tau) = \frac{1}{\Gamma(\gamma)}(t-\tau)^{\gamma-1}$. □

Remark 17. *In the particular case when the Lagrangian* F *of Corollary 16 does not depend on the fractional integral and the classical derivative, one obtains from* (10) *the Euler–Lagrange equation of [21].*

5 Generalized free-boundary variational problems

Assume now that in problem (3)–(4) the boundary conditions (4) are substituted by

$$y(a) \text{ is free and } y(b) = y_b. \quad (11)$$

Theorem 18. *If* y *is a solution to the problem of extremizing functional* (3) *with* (11) *as boundary conditions, then* y *satisfies the Euler–Lagrange equation* (5). *Moreover, the extra natural boundary condition*

$$\partial_3 F\left\{y\right\}^{\beta,\gamma}_{P_2,P_3}(a)k_\alpha(b,a) + K^{1-\beta}_{P_2^*}\left[\tau \mapsto \partial_4 F\left\{y\right\}^{\beta,\gamma}_{P_2,P_3}(\tau)k_\alpha(b,\tau)\right](a) = 0 \quad (12)$$

holds.

Proof. Under the boundary conditions (11), we do not require η in the proof of Theorem 14 to vanish at $t = a$. Therefore, following the proof of Theorem 14, we obtain

$$\begin{aligned}
&\partial_3 F \{y\}^{\beta,\gamma}_{P_2,P_3}(a) k_\alpha(b,a)\eta(a) + \eta(a) K^{1-\beta}_{P_2^*}\left[\tau \mapsto \partial_4 F\{y\}^{\beta,\gamma}_{P_2,P_3}(\tau)k_\alpha(b,\tau)\right](a) \\
&\quad + \int_a^b \eta(t)\Bigg(\partial_2 F\{y\}^{\beta,\gamma}_{P_2,P_3}(t)k_\alpha(b,t) - \frac{d}{dt}\left(\partial_3 F\{y\}^{\beta,\gamma}_{P_2,P_3}(t)k_\alpha(b,t)\right) \\
&\quad - A^{\beta}_{P_2^*}\left[\tau \mapsto \partial_4 F\{y\}^{\beta,\gamma}_{P_2,P_3}(\tau)k_\alpha(b,\tau)\right](t) + K^{\gamma}_{P_3^*}\left[\tau \mapsto \partial_5 F\{y\}^{\beta,\gamma}_{P_2,P_3}(\tau)k_\alpha(b,\tau)\right](t)\Bigg)dt = 0
\end{aligned} \tag{13}$$

for every admissible $\eta \in C^1([a,b];\mathbb{R})$ with $\eta(b) = 0$. In particular, condition (13) holds for those η that fulfill $\eta(a) = 0$. Hence, by the fundamental lemma of the calculus of variations, equation (5) is satisfied. Now, let us return to (13) and let η again be arbitrary at point $t = a$. Inserting (5), we obtain the natural boundary condition (12). □

Corollary 19. *Let $\mathcal{J}$ be the functional given by*

$$\mathcal{J}(y) = {}_aI^{\alpha}_b\left[t \mapsto F\left(t, y(t), {}^C_aD^{\beta}_t[y](t)\right)\right](b).$$

Let y be a minimizer of $\mathcal{J}$ satisfying the boundary condition $y(b) = y_b$. Then, y satisfies the Euler–Lagrange equation

$$(b-t)^{\alpha-1}\partial_2 F\left(t, y(t), {}^C_aD^{\alpha}_t[y](t)\right) + {}_tD^{\alpha}_b\left[\tau \mapsto (b-\tau)^{\alpha-1}\partial_3 F\left(\tau, y(\tau), {}^C_aD^{\beta}_\tau y(\tau)\right)\right](t) = 0 \tag{14}$$

and the natural boundary condition

$${}_aI^{1-\beta}_b\left[\tau \mapsto (b-\tau)^{\alpha-1}\partial_3 F\left(\tau, y(\tau), {}^C_aD^{\beta}_\tau y(\tau)\right)\right](a) = 0. \tag{15}$$

Proof. Let functional (3) be such that it does not depend on the classical (integer) derivative $y'(t)$ and on the K-op. If $P_2 =< a, t, b, 1, 0 >$, $h_{1-\beta}(t-\tau) = \frac{1}{\Gamma(1-\beta)}(t-\tau)^{-\beta}$, and $k_\alpha(x-t) = \frac{1}{\Gamma(\alpha)}(x-t)^{\alpha-1}$, then the B-op reduces to the left fractional Caputo derivative and we deduce (14) and (15) from (5) and (12), respectively. □

Corollary 20. *Let $\mathcal{J}$ be the functional given by*

$$\mathcal{J}(y) = \int_a^b F\left(t, y(t), y'(t), B^{\beta}_{P_2}[y](t), K^{\gamma}_{P_3}[y](t)\right)dt.$$

If y is a minimizer to $\mathcal{J}$ satisfying the boundary condition $y(b) = y_b$, then y satisfies the Euler–Lagrange equation

$$\partial_2 F\{y\}^{\beta,\gamma}_{P_2,P_3}(t) - \frac{d}{dt}\partial_3 F\{y\}^{\beta,\gamma}_{P_2,P_3}(t) - A^{\beta}_{P_2^*}\left[\partial_4 F\{y\}^{\beta,\gamma}_{P_2,P_3}\right](t) + K^{\gamma}_{P_3^*}\left[\partial_5 F\{y\}^{\beta,\gamma}_{P_2,P_3}\right](t) = 0 \tag{16}$$

and the natural boundary condition

$$\partial_3 F\{y\}^{\beta,\gamma}_{P_2,P_3}(a) + K^{1-\beta}_{P_2^*}\left[\partial_4 F\{y\}^{\beta,\gamma}_{P_2,P_3}\right](a) = 0. \tag{17}$$

Proof. Choose, in the problem defined by (3) and (11), $k_\alpha(x,t) \equiv 1$. Then, equations (16) and (17) follow from (5) and (12), respectively. □

6 Generalized isoperimetric problems

Let $\xi \in \mathbb{R}$. Among all functions $y : [a, b] \to \mathbb{R}$ satisfying boundary conditions

$$y(a) = y_a, \quad y(b) = y_b, \tag{18}$$

and an isoperimetric constraint of the form

$$\mathcal{I}(y) = K^{\alpha}_{P_1}\left[G\left\{y\right\}^{\beta,\gamma}_{P_2,P_3}\right](b) = \xi, \tag{19}$$

we look for the one that extremizes (i.e., minimizes or maximizes) a functional

$$\mathcal{J}(y) = K^{\alpha}_{P_1}\left[F\left\{y\right\}^{\beta,\gamma}_{P_2,P_3}\right](b). \tag{20}$$

Operators $K^{\alpha}_{P_1}$, $B^{\beta}_{P_2}$ and $K^{\gamma}_{P_3}$, as well as function F, are the same as in problem (3)–(4). Moreover, we assume that functional (19) satisfies hypotheses (H1)–(H4).

Definition 21. *A function $y : [a, b] \to \mathbb{R}$ is said to be* admissible *for problem* (18)–(20) *if functions $B^{\beta}_{P_2}[y]$ and $K^{\gamma}_{P_3}[y]$ exist and are continuous on $[a, b]$, and y satisfies the given boundary conditions* (18) *and the given isoperimetric constraint* (19)*.*

Definition 22. *An admissible function $y \in C^1([a, b], \mathbb{R})$ is said to be an* extremal *for $\mathcal{I}$ if it satisfies the Euler–Lagrange equation* (5) *associated with functional in* (19)*, i.e.,*

$$\begin{aligned} &k_\alpha(b,t)\partial_2 G\left\{y\right\}^{\beta,\gamma}_{P_2,P_3}(t) - \frac{d}{dt}\left(\partial_3 G\left\{y\right\}^{\beta,\gamma}_{P_2,P_3}(t)k_\alpha(b,t)\right)\\ &\quad - A^{\beta}_{P_2^*}\left[\tau \mapsto k_\alpha(b,\tau)\partial_4 G\left\{y\right\}^{\beta,\gamma}_{P_2,P_3}(\tau)\right](t) + K^{\gamma}_{P_3^*}\left[\tau \mapsto k_\alpha(b,\tau)\partial_5 G\left\{y\right\}^{\beta,\gamma}_{P_2,P_3}(\tau)\right](t) = 0, \end{aligned}$$

where $P_j^* =< a, t, b, q_j, p_j >$, $j = 2, 3$, and $t \in (a, b)$.

Theorem 23. *If y is a solution to the isoperimetric problem* (18)–(20) *and is not an extremal for $\mathcal{I}$, then there exists a real constant λ such that*

$$\begin{aligned} &k_\alpha(b,t)\partial_2 H\left\{y\right\}^{\beta,\gamma}_{P_2,P_3}(t) - \frac{d}{dt}\left(\partial_3 H\left\{y\right\}^{\beta,\gamma}_{P_2,P_3}(t)k_\alpha(b,t)\right)\\ &\quad - A^{\beta}_{P_2^*}\left[\tau \mapsto k_\alpha(b,\tau)\partial_4 H\left\{y\right\}^{\beta,\gamma}_{P_2,P_3}(\tau)\right](t) + K^{\gamma}_{P_3^*}\left[\tau \mapsto k_\alpha(b,\tau)\partial_5 H\left\{y\right\}^{\beta,\gamma}_{P_2,P_3}(\tau)\right](t) = 0 \end{aligned} \tag{21}$$

for all $t \in (a, b)$, where $H(t, y, u, v, w) = F(t, y, u, v, w) - \lambda G(t, y, u, v, w)$ and $P_j^* =< a, t, b, q_j, p_j >$.

Proof. Consider a two-parameter family of the form $\hat{y} = y + \varepsilon_1\eta_1 + \varepsilon_2\eta_2$, where for each $i \in \{1, 2\}$ we have $\eta_i(a) = \eta_i(b) = 0$. First we show that we can select $\varepsilon_2\eta_2$ such that $\hat{y}$ satisfies (19). Consider the quantity $\mathcal{I}(\hat{y}) = K^{\alpha}_{P_1}\left[G\left\{\hat{y}\right\}^{\beta,\gamma}_{P_2,P_3}\right](b)$. Looking to $\mathcal{I}(\hat{y})$ as a function of $\varepsilon_1, \varepsilon_2$, we define $\hat{I}(\varepsilon_1, \varepsilon_2) = \mathcal{I}(\hat{y}) - \xi$. Thus, $\hat{I}(0, 0) = 0$. On the other hand, applying integration by parts formulas (Theorems 9, 10 and 11), we obtain that

$$\begin{aligned} \left.\frac{\partial \hat{I}}{\partial \varepsilon_2}\right|_{(0,0)} &= \int_a^b \eta_2(t)\Bigg(k_\alpha(b,t)\partial_2 G\left\{y\right\}^{\beta,\gamma}_{P_2,P_3}(t) - \frac{d}{dt}\left(\partial_3 G\left\{y\right\}^{\beta,\gamma}_{P_2,P_3}(t)k_\alpha(b,t)\right)\\ &\quad - A^{\beta}_{P_2^*}\left[\tau \mapsto k_\alpha(b,\tau)\partial_4 G\left\{y\right\}^{\beta,\gamma}_{P_2,P_3}(\tau)\right](t) + K^{\gamma}_{P_3^*}\left[\tau \mapsto k_\alpha(b,\tau)\partial_5 G\left\{y\right\}^{\beta,\gamma}_{P_2,P_3}(\tau)\right](t)\Bigg)dt, \end{aligned}$$

where $P_j^* =< a, t, b, q_j, p_j >$, $j = 1, 2$. We assume that y is not an extremal for $\mathcal{I}$. Hence, the fundamental lemma of the calculus of variations implies that there exists a function η_2 such that

$\frac{\partial \hat{I}}{\partial \varepsilon_2}\Big|_{(0,0)} \neq 0$. According to the implicit function theorem, there exists a function $\varepsilon_2(\cdot)$ defined in a neighborhood of 0 such that $\hat{I}(\varepsilon_1, \varepsilon_2(\varepsilon_1)) = 0$. Let $\hat{J}(\varepsilon_1, \varepsilon_2) = \mathcal{J}(\hat{y})$. Function $\hat{J}$ has an extremum at $(0,0)$ subject to $\hat{I}(0,0) = 0$, and we have proved that $\nabla \hat{I}(0,0) \neq 0$. The Lagrange multiplier rule asserts that there exists a real number λ such that $\nabla(\hat{J}(0,0) - \lambda \hat{I}(0,0)) = 0$. Because

$$\frac{\partial \hat{J}}{\partial \varepsilon_1}\Big|_{(0,0)} = \int_a^b \Bigg(k_\alpha(b,t)\partial_2 F\{y\}^{\beta,\gamma}_{P_2,P_3}(t) - \frac{d}{dt}\left(\partial_3 F\{y\}^{\beta,\gamma}_{P_2,P_3}(t) k_\alpha(b,t)\right)$$
$$- A^{\beta}_{P_2^*}\left[\tau \mapsto k_\alpha(b,\tau)\partial_4 F\{y\}^{\beta,\gamma}_{P_2,P_3}(\tau)\right](t) + K^{\gamma}_{P_3^*}\left[\tau \mapsto k_\alpha(b,\tau)\partial_5 F\{y\}^{\beta,\gamma}_{P_2,P_3}(\tau)\right](t)\Bigg)\eta_1(t)dt$$

and

$$\frac{\partial \hat{I}}{\partial \varepsilon_1}\Big|_{(0,0)} = \int_a^b \Bigg(k_\alpha(b,t)\partial_2 G\{y\}^{\beta,\gamma}_{P_2,P_3}(t) - \frac{d}{dt}\left(\partial_3 G\{y\}^{\beta,\gamma}_{P_2,P_3}(t) k_\alpha(b,t)\right)$$
$$- A^{\beta}_{P_2^*}\left[\tau \mapsto k_\alpha(b,\tau)\partial_4 G\{y\}^{\beta,\gamma}_{P_2,P_3}(\tau)\right](t) + K^{\gamma}_{P_3^*}\left[\tau \mapsto k_\alpha(b,\tau)\partial_5 G\{y\}^{\beta,\gamma}_{P_2,P_3}(\tau)\right](t)\Bigg)\eta_1(t)dt,$$

one has

$$\int_a^b \Bigg\{ k_\alpha(b,t)\partial_2 F\{y\}^{\beta,\gamma}_{P_2,P_3}(t) - \frac{d}{dt}\left(\partial_3 F\{y\}^{\beta,\gamma}_{P_2,P_3}(t) k_\alpha(b,t)\right)$$
$$- A^{\beta}_{P_2^*}\left[\tau \mapsto k_\alpha(b,\tau)\partial_4 F\{y\}^{\beta,\gamma}_{P_2,P_3}(\tau)\right](t) + K^{\gamma}_{P_3^*}\left[\tau \mapsto k_\alpha(b,\tau)\partial_5 F\{y\}^{\beta,\gamma}_{P_2,P_3}(\tau)\right](t)$$
$$- \lambda\Bigg(k_\alpha(b,t)\partial_2 G\{y\}^{\beta,\gamma}_{P_2,P_3}(t) - \frac{d}{dt}\left(\partial_3 G\{y\}^{\beta,\gamma}_{P_2,P_3}(t) k_\alpha(b,t)\right)$$
$$- A^{\beta}_{P_2^*}\left[\tau \mapsto k_\alpha(b,\tau)\partial_4 G\{y\}^{\beta,\gamma}_{P_2,P_3}(\tau)\right](t) + K^{\gamma}_{P_3^*}\left[\tau \mapsto k_\alpha(b,\tau)\partial_5 G\{y\}^{\beta,\gamma}_{P_2,P_3}(\tau)\right](t)\Bigg)\Bigg\}\eta_1(t)dt = 0.$$

From the fundamental lemma of the calculus of variations (see, e.g., [22, Section 2.2]) it follows

$$k_\alpha(b,t)\partial_2 F\{y\}^{\beta,\gamma}_{P_2,P_3}(t) - \frac{d}{dt}\left(\partial_3 F\{y\}^{\beta,\gamma}_{P_2,P_3}(t) k_\alpha(b,t)\right)$$
$$- A^{\beta}_{P_2^*}\left[\tau \mapsto k_\alpha(b,\tau)\partial_4 F\{y\}^{\beta,\gamma}_{P_2,P_3}(\tau)\right](t) + K^{\gamma}_{P_3^*}\left[\tau \mapsto k_\alpha(b,\tau)\partial_5 F\{y\}^{\beta,\gamma}_{P_2,P_3}(\tau)\right](t)$$
$$- \lambda\Bigg(k_\alpha(b,t)\partial_2 G\{y\}^{\beta,\gamma}_{P_2,P_3}(t) - \frac{d}{dt}\left(\partial_3 G\{y\}^{\beta,\gamma}_{P_2,P_3}(t) k_\alpha(b,t)\right)$$
$$- A^{\beta}_{P_2^*}\left[\tau \mapsto k_\alpha(b,\tau)\partial_4 G\{y\}^{\beta,\gamma}_{P_2,P_3}(\tau)\right](t) + K^{\gamma}_{P_3^*}\left[\tau \mapsto k_\alpha(b,\tau)\partial_5 G\{y\}^{\beta,\gamma}_{P_2,P_3}(\tau)\right](t)\Bigg) = 0,$$

that is,

$$k_\alpha(b,t)\partial_2 H\{y\}^{\beta,\gamma}_{P_2,P_3}(t) - \frac{d}{dt}\left(\partial_3 H\{y\}^{\beta,\gamma}_{P_2,P_3}(t) k_\alpha(b,t)\right)$$
$$- A^{\beta}_{P_2^*}\left[\tau \mapsto k_\alpha(b,\tau)\partial_4 H\{y\}^{\beta,\gamma}_{P_2,P_3}(\tau)\right](t) + K^{\gamma}_{P_3^*}\left[\tau \mapsto k_\alpha(b,\tau)\partial_5 H\{y\}^{\beta,\gamma}_{P_2,P_3}(\tau)\right](t) = 0$$

with $H = F - \lambda G$. $\square$

Corollary 24. *Let y be a minimizer to the isoperimetric problem*

$$\mathcal{J}(y) = {}_aI_b^\alpha\left[t\mapsto F\left(t, y(t), {}_a^CD_t^\beta[y](t)\right)\right](b) \longrightarrow \min, \tag{22}$$

$$\mathcal{I}(y) = {}_aI_b^\alpha\left[t\mapsto G\left(t, y(t), {}_a^CD_t^\beta[y](t)\right)\right](b) = \xi, \tag{23}$$

$$y(a) = y_a, \quad y(b) = y_b. \tag{24}$$

If y is not an extremal of $\mathcal{I}$, then there exists a constant λ such that y satisfies

$$(b-t)^{\alpha-1}\partial_2 H\left(t, y(t), {}_a^CD_t^\alpha[y](t)\right) + {}_tD_b^\beta\left[\tau\mapsto (b-\tau)^{\alpha-1}\partial_3 H\left(\tau, y(\tau), {}_a^CD_\tau^\beta[y](\tau)\right)\right](t) = 0 \tag{25}$$

for all $t\in(a,b)$, where $H(t,y,v) = F(t,y,v) - \lambda G(t,y,v)$.

Proof. Let $k_\alpha(x,t) = \frac{1}{\Gamma(\alpha)}(x-t)^{\alpha-1}$, $h_{1-\beta}(t,\tau) = \frac{1}{\Gamma(1-\beta)}(t-\tau)^{-\beta}$, $P_1 =< a, b, b, 1, 0 >$ and $P_2 =< a, t, b, 1, 0 >$. Then the K-op and the B-op reduce to the left fractional integral and the left fractional Caputo derivative, respectively. Therefore, problem (22)–(24) is a particular case of problem (18)–(20), and (25) follows from (21) with $\partial_3 H = \partial_5 H = 0$. □

Corollary 25. *Let y be a minimizer to*

$$\mathcal{J}(y) = \int_a^b F\left(t, y(t), y'(t), B_{P_2}^\beta[y](t), K_{P_3}^\gamma[y](t)\right)dt \longrightarrow \min,$$

$$\mathcal{I}(y) = \int_a^b G\left(t, y(t), y'(t), B_{P_2}^\beta[y](t), K_{P_3}^\gamma[y](t)\right)dt = \xi,$$

$$y(a) = y_a, \; y(b) = y_b.$$

If y is not an extremal of $\mathcal{I}$, then there exists a constant λ such that y satisfies

$$\partial_2 H\left\{y\right\}_{P_2,P_3}^{\beta,\gamma}(t) - \frac{d}{dt}\partial_3 H\left\{y\right\}_{P_2,P_3}^{\beta,\gamma}(t) - A_{P_2^*}^\beta\left[\partial_4 H\left\{y\right\}_{P_2,P_3}^{\beta,\gamma}\right](t) + K_{P_3^*}^\gamma\left[\partial_5 H\left\{y\right\}_{P_2,P_3}^{\beta,\gamma}\right](t) = 0 \tag{26}$$

for all $t\in[a,b]$, where $H(t,y,u,v,w) = F(t,y,u,v,w) - \lambda G(t,y,u,v,w)$.

Proof. Let in problem (18)–(20) $P_1 =< a, b, b, 1, 0 >$ and kernel $k_\alpha(x,t)\equiv 1$. Then, the generalized fractional integral $K_{P_1}^\alpha$ becomes the classical integral and (26) follows from (21). □

7 Illustrative examples

We illustrate our results through two examples with different kernels: one of a fundamental problem (3)–(4) (Example 26), the other an isoperimetric problem (18)–(20) (Example 27).

Example 26. *Let $\alpha,\beta\in(0,1)$, $\xi\in\mathbb{R}$, $P_1 =< 0, 1, 1, 1, 0 >$, and $P_2 =< 0, t, 1, 1, 0 >$. Consider the following problem:*

$$\mathcal{J}(y) = K_{P_1}^\alpha\left[t\mapsto tK_{P_2}^\beta[y](t) + \sqrt{1-\left(K_{P_2}^\beta[y](t)\right)^2}\right](1) \longrightarrow \min,$$

$$y(0) = 1\,,\; y(1) = \frac{\sqrt{2}}{4} + \int_0^1 r_\beta(1-\tau)\frac{1}{(1+\tau^2)^{\frac{3}{2}}}d\tau,$$

with kernel h_β such that $h_\beta(t,\tau) = h_\beta(t-\tau)$ and $h_\beta(0)=1$. Here the resolvent $r_\beta(t)$ is related to the kernel $h_\beta(t)$ by $r_\beta(t) = \mathcal{L}^{-1}\left[s\mapsto \frac{1}{s\widetilde{h}_\beta(s)} - 1\right](t)$, $\widetilde{h}_\beta(s) = \mathcal{L}\left[t\mapsto h_\beta(t)\right](s)$, where $\mathcal{L}$ and $\mathcal{L}^{-1}$ are the direct and the inverse Laplace operators, respectively. We apply Theorem 14 with Lagrangian F given by $F(t,y,u,v,w) = tw + \sqrt{1-w^2}$. Because

$$y(t) = \frac{1}{(1+t^2)^{\frac{3}{2}}} + \int_0^t r_\beta(t-\tau)\frac{1}{(1+\tau^2)^{\frac{3}{2}}}d\tau$$

is the solution to the Volterra integral equation of first kind (see, e.g., Equation 16, p. 114 of [47])

$$K_{P_2}^{\beta}[y](t) = \frac{t\sqrt{1+t^2}}{1+t^2},$$

it satisfies our generalized Euler–Lagrange equation (5)*, i.e.,*

$$K_{P_2^*}^{\beta}\left[\tau \mapsto k_\alpha(b,\tau)\left(\frac{-K_{P_2^*}^{\beta}[y](\tau)}{\sqrt{1-\left(K_{P_2^*}^{\beta}[y](\tau)\right)^2}}+\tau\right)\right](t) = 0.$$

In particular, for the kernel $h_\beta(t-\tau) = \cosh(\beta(t-\tau))$, the boundary conditions are $y(0) = 1$ and $y(1) = 1+\beta^2(1-\sqrt{2})$, and the solution is $y(t) = \frac{1}{(1+t^2)^{\frac{3}{2}}} + \beta^2\left(1-\sqrt{1+t^2}\right)$ (cf. [47, p. 22]).

In the next example we make use of the Mittag–Leffler function of two parameters: if $\alpha, \beta > 0$, then the Mittag–Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}\,.$$

This function appears naturally in the solution of fractional differential equations, as a generalization of the exponential function [27].

Example 27. *Let $\alpha, \beta \in (0,1)$, $\xi \in \mathbb{R}$, and $\xi \notin \left\{\pm\frac{1}{4}\right\}$. Consider the following problem:*

$$\begin{gathered}
\mathcal{J}(y) = {}_0I_1^{\alpha}\left[\sqrt{1+\left(y' + {}_0^C D_t^{\beta}[y]\right)^2}\right](1) \longrightarrow \min, \\
\mathcal{I}(y) = {}_0I_1^{\alpha}\left[\left(y' + {}_0^C D_t^{\beta}[y]\right)^2\right](1) = \xi, \\
y(0) = 0\,,\ y(1) = \int_0^1 E_{1-\beta,1}\left(-(1-\tau)^{1-\beta}\right)\frac{\sqrt{1-16\xi^2}}{4\xi}d\tau,
\end{gathered} \tag{27}$$

which is an example of (18)*–*(20) *with p-sets $P_1 =< 0,1,1,1,0 >$ and $P_2 =< 0,t,1,1,0 >$ and kernels $k_\alpha(x-t) = \frac{1}{\Gamma(\alpha)}(x-t)^{\alpha-1}$ and $h_{1-\beta}(t-\tau) = \frac{1}{\Gamma(1-\beta)}(t-\tau)^{-\beta}$. Function H of Theorem 23 is given by $H(t,y,u,v,w) = \sqrt{1+(u+v)^2} - \lambda(u+v)^2$. One can easily check (see [27, p. 324]) that*

$$y(t) = \int_0^t E_{1-\beta,1}\left(-(t-\tau)^{1-\beta}\right)\frac{\sqrt{1-16\xi^2}}{4\xi}d\tau \tag{28}$$

- *is not an extremal for $\mathcal{I}$;*
- *satisfies $y' + {}_0^C D_t^{\beta}[y] = \frac{\sqrt{1-16\xi^2}}{4\xi}$.*

Moreover, (28) *satisfies* (21) *for $\lambda = 2\xi$, i.e.,*

$$\begin{aligned}
&-\frac{d}{dt}\left((1-t)^{\alpha-1}\left(y'(t) + {}_0^C D_t^{\beta}[y](t)\right)\left(\frac{1}{\sqrt{1+\left(y'(t)+{}_0^C D_t^{\beta}[y](t)\right)^2}} - 4\xi\right)\right) \\
&+ {}_tD_1^{\beta}\left[\tau \mapsto (1-\tau)^{\alpha-1}\left(y'(\tau) + {}_0^C D_\tau^{\beta}[y](\tau)\right)\left(\frac{1}{\sqrt{1+\left(y'(\tau)+{}_0^C D_\tau^{\beta}[y](\tau)\right)^2}} - 4\xi\right)\right](t) = 0
\end{aligned}$$

for all $t \in (0,1)$. We conclude that (28) *is an extremal for problem* (27).

8 Applications to Physics

If the functional (3) does not depend on B-op and K-op, then Theorem 14 gives the following result: if y is a solution to the problem of extremizing

$$\mathcal{J}(y) = \int_a^b F\left(t, y(t), y'(t)\right) k_\alpha(b,t) dt \tag{29}$$

subject to $y(a) = y_a$ and $y(b) = y_b$, where $\alpha \in (0,1)$, then

$$\partial_2 F\left(t, y(t), y'(t)\right) - \frac{d}{dt}\partial_3 F\left(t, y(t), y'(t)\right) = \frac{1}{k_\alpha(b,t)} \cdot \frac{d}{dt} k_\alpha(b,t) \partial_3 F\left(t, y(t), y'(t)\right). \tag{30}$$

We recognize on the right hand side of (30) the generalized weak dissipative parameter

$$\delta(t) = \frac{1}{k_\alpha(b,t)} \cdot \frac{d}{dt} k_\alpha(b,t).$$

8.1 Quantum mechanics of the damped harmonic oscillator

As a first application, let us consider kernel $k_\alpha(b,t) = \mathrm{e}^{\alpha(b-t)}$ and the Lagrangian

$$L\left(y, \dot{y}\right) = \frac{1}{2} m \dot{y}^2 - V(y),$$

where $V(y)$ is the potential energy and m stands for mass. The Euler–Lagrange equation (30) gives the following second order ordinary differential equation:

$$\ddot{y}(t) - \alpha \dot{y}(t) = -\frac{1}{m} V'(y(t)). \tag{31}$$

Equation (31) coincides with (14) of [24], obtained by modification of Hamilton's principle.

8.2 Fractional Action-Like Variational Approach (FALVA)

We now extend some of the recent results of [15–18], where the fractional action-like variational approach (FALVA) was proposed to model dynamical systems. FALVA functionals are particular cases of (29), where the fractional time integral introduces only one parameter α. Let us consider the Caldirola–Kanai Lagrangian [8,16,18]

$$L\left(t, y, \dot{y}\right) = m(t) \left(\frac{\dot{y}^2}{2} - \omega^2 \frac{y^2}{2} \right), \tag{32}$$

which describes a dynamical oscillatory system with exponentially increasing time dependent mass, where ω is the frequency and $m(t) = m_0 \mathrm{e}^{-\gamma b} \mathrm{e}^{\gamma t} = \bar{m}_0 \mathrm{e}^{\gamma t}$, $\bar{m}_0 = m_0 \mathrm{e}^{-\gamma b}$. Using our generalized FALVA Euler–Lagrange equation (30) with kernel $k_\alpha(b,t)$ to Lagrangian (32), we obtain

$$\ddot{y}(t) + \left(\delta(t) + \gamma\right) \dot{y}(t) + \omega^2 y(t) = 0. \tag{33}$$

We study two particular kernels.

1. If we choose kernel

$$k_\alpha(b,t) = \frac{(\rho+1)^{1-\alpha}}{\Gamma(\alpha)} \left(b^{\rho+1} - t^{\rho+1}\right) t^\rho, \tag{34}$$

defined in [26], then the Euler–Lagrange equation is

$$\partial_2 F\left(t, y(t), y'(t)\right) - \frac{d}{dt}\partial_3 F\left(t, y(t), y'(t)\right) = \left(\frac{(1-\alpha)(\rho+1)t^\rho}{b^{\rho+1} - t^{\rho+1}} \right) \partial_3 F\left(t, y(t), y'(t)\right). \tag{35}$$

In particular, when $\rho \to 0$, (34) becomes the kernel of the Riemann–Liouville fractional integral, and equation (35) gives

$$\partial_2 F\left(t, y(t), y'(t)\right) - \frac{d}{dt}\partial_3 F\left(t, y(t), y'(t)\right) = \frac{1-\alpha}{b-t}\partial_3 F\left(t, y(t), y'(t)\right),$$

which is the Euler–Lagrange equation proved in [16]. For $\rho \neq 0$, we have

$$\delta(t) = \frac{(1-\alpha)(\rho+1)t^{\rho}}{b^{\rho+1} - t^{\rho+1}} \to 0 \text{ if } t \to \infty \text{ or } t \to 0.$$

Therefore, both at the very early time and at very large time, dissipation disappears. Moreover, if $\rho \to 0$, then

$$\delta(t) = \frac{1-\alpha}{b-t} \to \begin{cases} 0 & \text{if } t \to \infty \\ \frac{1-\alpha}{b} & \text{if } t \to 0. \end{cases}$$

This shows that at the origin of time, the time-dependent dissipation becomes stationary, and that at very large time no dissipation, of any kind, exists.

2. If we choose kernel $k_\alpha(b,t) = (\cosh b - \cosh t)^{\alpha-1}$, then

$$\partial_2 F\left(t, y(t), y'(t)\right) - \frac{d}{dt}\partial_3 F\left(t, y(t), y'(t)\right) = -(\alpha-1)\frac{\sinh t}{\cosh b - \cosh t}\partial_3 F\left(t, y(t), y'(t)\right) \quad (36)$$

and

$$\delta(t) = -(\alpha-1)\frac{\sinh t}{\cosh b - \cosh t} \to \begin{cases} \alpha - 1 & \text{if } t \to \infty \\ 0 & \text{if } t \to 0. \end{cases}$$

In contrast with previous case, item 1, here dissipation does not disappear at late-time dynamics.

We note that there is a small inconsistence in [16], regarding to the coefficient of $\dot{y}(t)$ in (33), and a small inconsistence in [18], regarding a sign of (36).

9 Conclusion

In this article we unify, subsume and significantly extend the necessary optimality conditions available in the literature of the fractional calculus of variations. It should be mentioned, however, that since fractional operators are nonlocal, it can be extremely challenging to find analytical solutions to fractional problems of the calculus of variations and, in many cases, solutions may not exist. In our paper we give two examples with analytic solutions, and many more can be found borrowing different kernels from the book [47]. On the other hand, one can easily choose examples for which the fractional Euler–Lagrange differential equations are hard to solve, and in that case one needs to use numerical methods [2,6,48,49]. The question of existence of solutions to fractional variational problems is a complete open area of research. This needs attention. Indeed, in the absence of existence, the necessary conditions for extremality are vacuous: one cannot characterize an entity that does not exist in the first place. For solving a problem of the fractional calculus of variations one should proceed along the following three steps: (i) first, prove that a solution to the problem exists; (ii) second, verify the applicability of necessary optimality conditions; (iii) finally, apply the necessary conditions which identify the extremals (the candidates). Further elimination, if necessary, identifies the minimizer(s) of the problem. All three steps in the above procedure are crucial. As mentioned by Young in [59], the calculus of variations has born from the study of necessary optimality conditions, but any such theory is "naive" until the existence of minimizers is verified. The process leading to the existence theorems was introduced by Leonida Tonelli in 1915 by the so-called direct method [56]. During two centuries, mathematicians were developing "the naive approach to the calculus of variations". There was, of course, good reasons

why the existence problem was only solved in the beginning of XX century, two hundred years after necessary optimality conditions began to be studied: see [13, 57] and references therein. Similar situation happens now with the fractional calculus of variations: the subject is only fifteen years old, and is still in the "naive period". We believe time has come to address the existence question, and this will be considered in a forthcoming paper.

Acknowledgements

This work was supported by *FEDER* funds through *COMPETE* — Operational Programme Factors of Competitiveness ("Programa Operacional Factores de Competitividade") and by Portuguese funds through the *Center for Research and Development in Mathematics and Applications* (University of Aveiro) and the Portuguese Foundation for Science and Technology ("FCT — Fundação para a Ciência e a Tecnologia"), within project PEst-C/MAT/UI4106/2011 with COMPETE number FCOMP-01-0124-FEDER-022690. Odzijewicz was also supported by FCT through the Ph.D. fellowship SFRH/BD/33865/2009; Malinowska by Białystok University of Technology grant S/WI/02/2011; and Torres by FCT through the Portugal–Austin (USA) cooperation project UTAustin/MAT/0057/2008.

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